LINKÖPING UNIVERSITY

Advanced Computational Statistics – Spring 2025
Assignment for Lecture 7

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Perform the solutions individually and send your report **until June 7** by email to me. Try to keep this deadline. You will then be assigned to perform a peer-review which you have to hand in until June 30. However, if you have problems with the June-7-deadline, there will be a final deadline on September 30 for all assignments. Please send me one **pdf-file** with your report, and additionally, please send me your code in one separate **plain-text file**. **The following two problems are mandatory.**

Problem 7.1

We consider a case when the prior distribution for the mean μ in a normal distribution has the following Laplace density:

$$f_{\text{prior}}(\mu) = \frac{1}{2}e^{-|\mu|}.$$

When normally distributed data is observed, the posterior $f_{\text{posterior}}(\mu)$ has a density proportional to:

$$g(\mu|\bar{x}) = f_{\text{prior}}(\mu) \cdot e^{-(\mu-\bar{x})^2/(2\tau^2)},$$

where $\bar{x}$ is the observed mean and $\tau^2 = \sigma^2/n$ is the variance of the observed mean. Assume $\tau^2 = 1$ and that the observed mean was $\bar{x} = 2.04$. It is of interest here to compute the probability $P(\mu \geq 1) = \int_1^\infty f_{\text{posterior}}(\mu)\, d\mu$ based on the posterior distribution using several techniques as required below.

a. Determine the proportionality constant of the posterior density (you can use a build-in function for integration for this part) and plot it.

b. Write an own program to compute $\int_1^\infty f_{\text{posterior}}(\mu)\, d\mu$ with Simpson's rule where the number of nodes is iteratively increased until a stop criterion is met. For this, you can truncate the integral at a suitably chosen upper bound. Compute the integral.

c. Write an own program using rejection sampling with an envelope of your choice to simulate draws of the posterior distribution. Compute a Monte Carlo estimate for $P(\mu \geq 1)$.

d. Write an own program using a Metropolis algorithm to generate draws of the posterior. Compute a Monte Carlo estimate for $P(\mu \geq 1)$.

Problem 7.2

Independent observations X in the interval $[0, 1]$ are obtained from a technical process. They are distributed according to following probability density:

$$f(x) = \frac{2}{1 + e^{-10x+5}} \cdot \mathbf{1}\{0 \leq x \leq 1\},$$

where $\mathbf{1}\{\dots\}$ is the indicator function (which is 1 if the following condition is fulfilled and is 0 otherwise). It is here of interest to quantify the probability for observations which are smaller than 0.1, $P(X < 0.1)$, using Monte Carlo simulations.

a. Plot the function f.

b. Choose two different sampling functions g for importance sampling which both have the potential to be good in reducing the variance of the Monte Carlo estimator.

c. After motivating and deciding on a number of repetitions, compute the importance sampling estimator (with non-standardized weights) for $P(X < 0.1)$ and quantify the standard error of the estimator for both sampling functions. How much improvement gives this compared with the standard error from the usual Monte Carlo estimator?

d. Repeat part c. with the importance sampling estimator with standardized weights and compare the standard errors. Estimate also the bias of the estimates.